Set theory is the theory of everything

6/13/2021

Introduction of the topic for this summer

Reference: Kunen, The Foundations of Mathematics

0.3 Why Read This Book?

This book describes the basics of set theory, model theory, proof theory, and recursion theory; these are all parts of what is called mathematical logic. There are three reasons one might want to read about this:

1. As an introduction to logic.
2. For its applications in topology, analysis, algebra, AI, databases.
3. Because the foundations of mathematics is relevant to philosophy.

1. If you plan to become a logician, then you will need this material to understand more advanced work in the subject.

2. Set theory is useful in any area of math dealing with uncountable sets; model theory is closely related to algebra. Questions about decidability come up frequently in math and computer science. Also, areas in computer science such as artificial intelligence and databases often use notions from model theory and proof theory.

3. The title of this book is "The Foundations of Mathematics", and there are a number of philosophical questions about this subject. Whether or not you are interested in the philosophy, it is a good way to tie together the various topics, so we'll begin with that. Further philosophical remarks occur in Subsection I.7.2 and Chapter III.

What is the Foundations of Math?

- it involves the axiomatic method → write down axioms and prove theorems from the axioms

Examples of axiomatic method:

1) Geometry → axioms developed by Ancient Greeks
→ two distinct points determine a line

2) Group theory

a group $(G, \cdot)$

$GP = \{ \gamma_1, \gamma_2 \}$

$\gamma_1 : \forall x, y, z \, [x \cdot (y \cdot z) = (x \cdot y) \cdot z]$

$\gamma_2 : \exists u \big[\forall x \, [x \cdot u = u \cdot x = x] \wedge \forall x \, \exists y \, [x \cdot y = y x = u] \big]$

What does this mean? G is a set and $\cdot$ is a function $G \times G \to G$ s.t. γ_1 and γ_2 hold in G.

We say G is our <u>domain of discourse</u> or <u>universe</u>.

From axioms one proves theorems

$$GP \vdash \forall x y z \, [xy = xz \rightarrow y = z]$$

3) Set theory • for infinite sets → Cantor (1880s)
worked mainly not axiomatically and was aware this could lead to contradictions

• ZFC system developed later now the "standard" axioms of set theory

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But what is a proof?

The Foundations of Math should give a precise definition of what a mathematical statement is and what a proof is

→ Model theory and Proof Theory

GP: a set of two sentences in predicate logic

$\underbrace{\text{formal proof: finite sequence of sentences in this formal language}}_{\text{syntax}}$

groups: meaning, structures → semantics

Recursion theory → theory of algorithms and computability

The following sets are decidable

1) The set of primes
2) The set of axioms of ZFC
3) The set of valid C programs

The following are not decidable:

4) The set of C programs that eventually halt
5) $\{\varphi : ZFC \vdash \varphi\}$

"mathematical truth is not decidable"

Is ZFC consistent (i.e. doesn't prove $\varphi \wedge \neg\varphi$)?

→ Philosophical discussion Ch III

Gödel's Second Incompleteness Theorem → implies consistency of ZFC cannot be proved by "elementary means".

Set Theory

The axioms:

Axiom 0. Set Existence.

"the universe is nonempty"

$$\exists x(x = x)$$

Axiom 1. Extensionality.

"two sets with the same elements are the same"

$$\forall z(z \in x \leftrightarrow z \in y) \quad \rightarrow \quad x = y$$

Axiom 2. Foundation.

"every non-empty set is disjoint from one of its elements"

$$\exists y(y \in x) \quad \rightarrow \quad \exists y(y \in x \wedge \neg \exists z(z \in x \wedge z \in y))$$

Axiom 3. Comprehension Scheme. For each formula, φ, without y free,

$$\exists y \forall x(x \in y \quad \leftrightarrow \quad x \in z \wedge \varphi(x))$$

Axiom 4. Pairing.

$$\exists z(x \in z \wedge y \in z)$$

"$\exists$ z that contains both x and y"

Axiom 5. Union.

$$\exists A \forall Y \forall x(x \in Y \wedge Y \in \mathcal{F} \quad \rightarrow \quad x \in A)$$

"$\bigcup_{Y \in \mathcal{F}} Y = A$"

Axiom 6. Replacement Scheme. For each formula, φ, without B free,

$$\forall x \in A \, \exists ! y \, \varphi(x,y) \quad \rightarrow \quad \exists B \, \forall x \in A \, \exists y \in B \, \varphi(x,y)$$

The rest of the axioms are a little easier to state using some defined notions. On the basis of Axioms 1,3,4,5, define $\subseteq$ (subset), $\emptyset$ (or 0; empty set), S (ordinal successor function), $\cap$ (intersection), and $\mathrm{SING}(x)$ (x is a singleton) by:

$$\begin{array}{rcl} x \subseteq y & \iff & \forall z(z \in x \rightarrow z \in y) \\ x = \emptyset & \iff & \forall z(z \notin x) \\ y = S(x) & \iff & \forall z(z \in y \leftrightarrow z \in x \vee z = x) \\ w = x \cap y & \iff & \forall z(z \in w \leftrightarrow z \in x \wedge z \in y) \\ \mathrm{SING}(x) & \iff & \exists y \in x \, \forall z \in x (z = y) \end{array}$$

Axiom 7. Infinity.

$$\exists x \big(\emptyset \in x \wedge \forall y \in x (S(y) \in x) \big)$$

Axiom 8. Power Set.

$$\exists y \forall z (z \subseteq x \rightarrow z \in y)$$

Axiom 9. Choice.

$$\emptyset \notin F \wedge \forall x \in F \, \forall y \in F (x \neq y \rightarrow x \cap y = \emptyset) \quad \rightarrow \quad \exists C \, \forall x \in F (\mathrm{SING}(C \cap x))$$

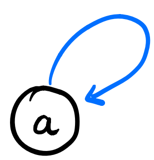
- ☛ ZFC = Axioms 1–9. ZF = Axioms 1–8.
- ☛ ZC and Z are ZFC and ZF, respectively, with Axiom 6 (Replacement) deleted.
- ☛ Z^-, ZF^-, ZC^-,ZFC^- are Z, ZF, ZC, ZFC, respectively, with Axiom 2 (Foundation) deleted.

So what do these mean?

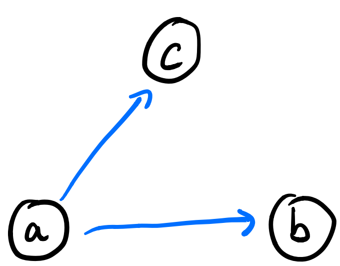
<u>Example</u> Which of the axioms 1, 2, 4 are true of binary relation E on domain D?

1. $D = \{a\}$, $E = \{(a,a)\}$

1	2	4
✓	✗	✓



2. $D = \{a,b,c\}$, $E = \{(a,b), (a,c)\}$



1	2	4
✗	✓	✗

So why is set theory the theory of everything?

→ All abstract mathematical concepts are set-theoretic.

→ All concrete math objects are specific sets.

For example, a function is a set of ordered pairs s.t.

$$\forall x, y, z \ [(x,y) \in f \wedge (x,z) \in f \rightarrow y = z]$$

But what is an ordered pair → something uniquely associated to x and y. Formally

$$(x,y) = \{\{x\}, \{x,y\}\}$$

You can show

$$ZFC \vdash \forall x,y,x',y' \; [(x,y)=(x',y') \rightarrow x=x' \wedge y=y']$$

But how do you define a number? What is 2?

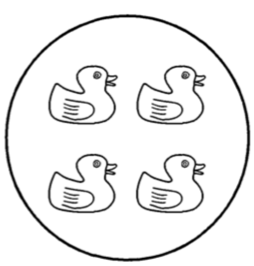
We could define it as a specific set with 2 elements in it:

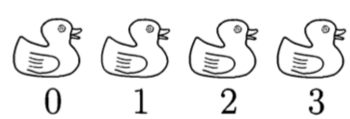
$$\{0,1\}$$

But what is 0? $\quad 0=\emptyset \quad$ and $\quad 1=\{\emptyset\}$

So, we can define $\mathbb{N}=\omega=\{0,1,2,\dots\}$

Counting





$$|D|=4$$

To count $\mathbb{Q}$ you can use $\mathbb{N} = \omega$.

$$\begin{matrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \\ \frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} \\ \\ \frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} \end{matrix}$$

$|\mathbb{Q}| = \omega = \aleph_0$ (first infinite cardinal)

But $\mathbb{R}$ is uncountable. If you take any list of reals $\{a_0, a_1, \dots\} = L_\omega$ we can't get the whole $\mathbb{R}$.

<u>Idea</u> You can keep counting by looking at $\mathbb{R} \setminus L_\omega$ and picking an element

$L_{\omega+1} = \{a_0, \dots\dots\} \cup \{a_\omega\}$

So, we have $0, 1, \dots, \omega, \omega+1, \omega+2, \dots\dots, \omega_1$

first uncountable ordinal

If you use enough of these you can get $\mathbb{R}$.

Extensionality

Axiom 1. Extensionality.

$$\forall z(z \in x \leftrightarrow z \in y) \quad \rightarrow \quad x = y$$

This says that if two sets have the same elements then they are the same set.

But this says something more

→ everything in the universe we are considering must be a set

Why? If we allow objects in our universe say a duck (D) and a pig (P) then the axiom will not hold:

$\forall z \in P$ is false

$z \in D$ is false

So, $z \in P \leftrightarrow z \in D$ is true but P and D are different.

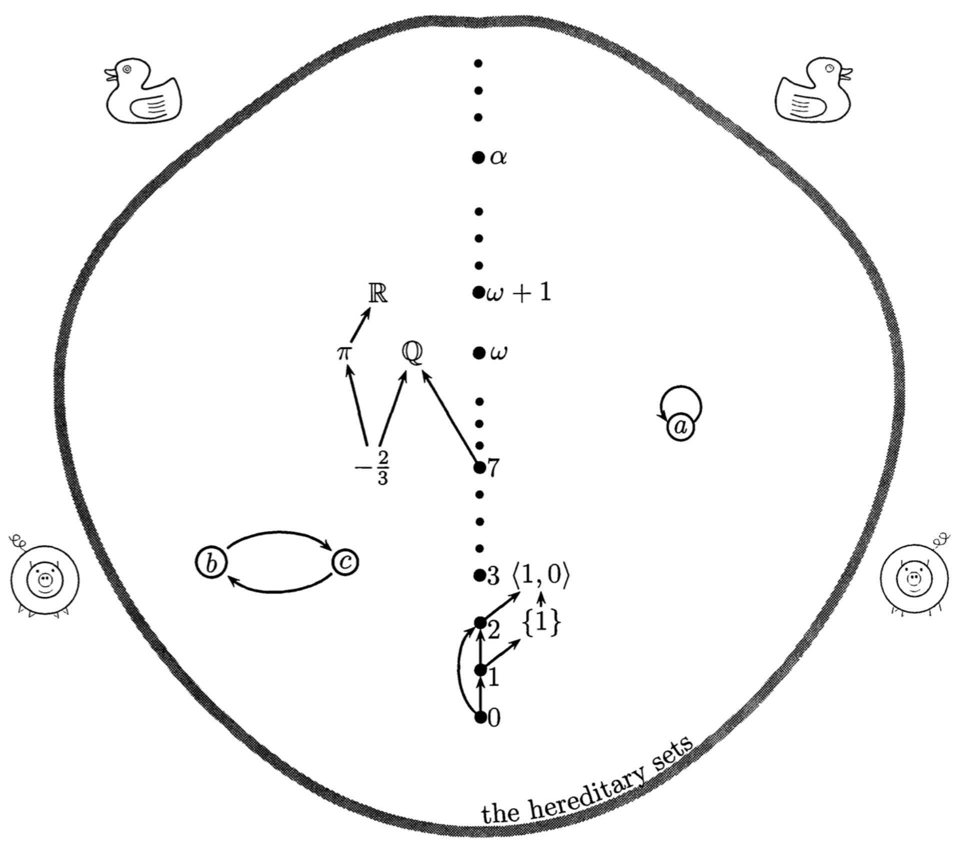
Moreover, sets of objects are also not in our universe

$\forall z$ in universe, $z \in \{P\} \leftrightarrow z \in \{D\}$ is true but $\{P\} \neq \{D\}$

What this means is that if x is in our universe then its elements are sets too and elements of elements and so on

$\rightarrow$ x is hereditarily a set

Set theoretic universe in ZF^-



$a \to b$ means $a \in b$

Theorem There is at most one empty set.

Proof Suppose there are two denoted x and y. Then $\forall z, z \in x$ is false and $\forall z, z \in y$ is false.

$\Rightarrow \forall z \, (z \in x \Leftrightarrow z \in y)$ is true

$\overset{\text{Extensionality}}{\Rightarrow}$ $x = y$. So, if it exists the empty set is unique.

In the next talk we'll see a discussion of existence of $\emptyset$.